[Proc. Roy. Soc. Victoria, 41 (N.S.), Pt. II., 1929.]

Art. XI.—*Solitary Waves at the Common Boundary of two liquids.*

By FRANCES E. ALLAN, M.A.

(Communicated by J. H. Michell).

[Read 11th October, 1928; issued separately 3rd April, 1929.]

The form and the velocity of solitary, or indefinitely long, waves in a single liquid have been examined experimentally by Scott Russell and mathematically by Boussinesq and Rayleigh. The much wider problem of the possible aperiodic wave forms at the common boundary of two superposed liquids does not seem to have received similar treatment. Those who have treated the subject of waves of finite height at the surface of separation of two liquids have dealt rather with the case of periodic waves, for which a different method is suitable. (Priestly, *Camb. Phil. Soc. Proc.*, 1910; Lamb, *ibid.*, 1922; Kolchine, *Math. Ann.*, 1927-8.)

The discussion here given follows the method used by J. H. Michell in unpublished work.

The motion is supposed two-dimensional, and will be treated as steady by choice of an origin of coordinates moving at the rate of the wave-form. The axis of x is taken horizontal and the axis of y directed upwards. The independent variables are changed from x, y to x, ψ where ψ is the stream function for the motion. This simplifies the treatment of the conditions over the boundaries, the coordinate ψ being constant over each of them. The dependent variable to be found in terms of x and ψ is now y, for which, therefore, a differential equation must be found. When y is found the form of a boundary is given in Cartesian coordinates by ascribing the corresponding constant value to ψ.

In carrying out the process of approximation we take as the general mathematical characteristic of the long-wave motion that the variation of a quantity specifying it (in particular, the gradient of the wave form), in a distance equal to the depth of either liquid, is a small fraction of the quantity itself. Thus, if we take the unit of length as of the order of magnitude of the depth of either liquid, the second derivative d^2y/dx^2 is to be a small fraction of dy/dx, and so for higher derivatives. The assumption is to include the smallness of dy/dx itself. The general discussion terminates in the expression of the gradient dy/dx of the wave form in terms of y. I have considered the conditions under which the gradient takes the factor form appropriate to either a crested or an inverted (trough) wave form. The expression of x in terms of y in general involves elliptic integrals of the third kind. Where the undisturbed depth of the lower liquid is small we may find an approximate equation involving an elliptic integral of the first kind only to determine the form of the symmetric wave. I have dealt, finally, with a case of asymmetric wave (bore) where the gradient-equation for the form can be integrated without further approximation.

The Differential Equation for y.

In terms of independent variables x, y, the corresponding components of velocity are given by

$$u = -\partial\psi/\partial y,$$
$$v = \partial\psi/\partial x,$$

and the vorticity by

$$\omega = \partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2.$$

When the independent variables x, ψ, are introduced we have

$$v = \left(\frac{\partial\psi}{\partial x}\right)_{y \text{ const.}}$$

$$= \frac{-\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial \psi}}, \text{when } y \text{ is a function of } x \text{ and } \psi,$$

and

$$u = -\left(\frac{\partial\psi}{\partial y}\right)_{x \text{ const.}}$$

$$= -\frac{1}{\dfrac{\partial y}{\partial\psi}}, \text{when } \psi \text{ and } x \text{ are the independent variables.}$$

Also

$$\left(\frac{\partial v}{\partial x}\right)_{y \text{ const.}} = \left(\frac{\partial v}{\partial x}\right)_{\psi \text{ const.}} - \frac{\dfrac{\partial v}{\partial\psi}\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial\psi}},$$

and

$$\left(\frac{\partial u}{\partial y}\right)_{x \text{ const.}} = \frac{\dfrac{\partial u}{\partial\psi}}{\dfrac{\partial y}{\partial\psi}}.$$

Therefore, as a function of x and ψ,

$$\omega = \frac{\partial v}{\partial x} - \frac{\dfrac{\partial v}{\partial\psi}\dfrac{\partial y}{\partial x}}{\dfrac{\partial y}{\partial\psi}} - \frac{\dfrac{\partial u}{\partial\psi}}{\dfrac{\partial y}{\partial\psi}}.$$

Whence, substituting for u and v,

$$\omega = \left\{-\frac{\dfrac{\partial^2 y}{\partial x^2}}{\dfrac{\partial y}{\partial\psi}} + \frac{\dfrac{\partial y}{\partial x}\dfrac{\partial^2 y}{\partial x\partial\psi}}{\left(\dfrac{\partial y}{\partial\psi}\right)^2}\right\} + \left\{\frac{\dfrac{\partial^2 y}{\partial x\partial\psi}\dfrac{\partial y}{\partial x}}{\left(\dfrac{\partial y}{\partial\psi}\right)^2} - \frac{\dfrac{\partial^2 y}{\partial\psi^2}\left(\dfrac{\partial y}{\partial x}\right)^2}{\left(\dfrac{\partial y}{\partial\psi}\right)^3}\right\} - \frac{\dfrac{\partial^2 y}{\partial\psi^2}}{\left(\dfrac{\partial y}{\partial\psi}\right)^3},$$

that is,

$$-\omega\left(\frac{\partial y}{\partial \psi}\right)^3=\frac{\partial^2 y}{\partial x^2}\left(\frac{\partial y}{\partial \psi}\right)^2-2\frac{\partial y}{\partial x}\frac{\partial y}{\partial \psi}\frac{\partial^2 y}{\partial x \partial \psi}+\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}\frac{\partial^2 y}{\partial \psi^2}.$$

Therefore for irrotational motion, where $\omega=0$, we have

$$\frac{\partial^2 y}{\partial x^2}\left(\frac{\partial y}{\partial \psi}\right)^2-2\frac{\partial y}{\partial x}\frac{\partial y}{\partial \psi}\frac{\partial^2 y}{\partial x \partial \psi}+\left\{1+\left(\frac{\partial y}{\partial x}\right)^2\right\}\frac{\partial^2 y}{\partial \psi^2}=0. \quad (1)$$

To investigate a type of irrotational waves we must now find an approximate solution of this equation which will satisfy also the boundary conditions of the problem.

J. H. Michell has used this process as an alternative method of determining the well-known results for the infinitesimal and solitary long waves at the free upper surface of a liquid. The method applies equally well to problems on superposed liquids, and I have used it to find the equation to the form of the wave of finite height and wave length as far as the terms of the sixth order in the wave height.

The question to be considered here, however, is the form of the long wave at the boundary between two liquids in relative motion, the whole being confined between parallel planes at a distance h apart.

Let $y=0$, $y=h$ be the fixed horizontal planes between which the liquids lie. Let $\psi=0$ at $y=0$, $\psi=a$ at $y=h$ and $\psi=c$ at the interface of the liquids. Finally, let ρ, ρ' be the densities of the lower and upper liquids and U, V their respective "undisturbed" velocities.

At the first step in the approximate solution of the differential equation (1) for y, we neglect the first two terms as of the second order and the equation then reduces to

$$\frac{\partial^2 y}{\partial \psi^2}=0. \quad \text{.................................}(2)$$

On integration, this gives, for the lower liquid,

$$y=\eta\psi, \quad \text{.................................}(3)$$

where η is a function of x. (There is no term independent of ψ since $y=0$ when $\psi=0$.)

Substituting the value of y given by (3) in the second order terms of (1), and integrating again, we find

$$y+\tfrac{1}{6}\psi^3\left(\eta^2\frac{\partial^2 \eta}{\partial x^2}-2\eta\left.\overline{\frac{\partial \eta}{\partial x}}\right|^2\right)=\eta\psi, \quad \text{.....................}(4)$$

and putting $y=\eta\psi$ in the second order terms of (4), we obtain

$$y+\tfrac{1}{6}\left(y^2\frac{\partial^2 y}{\partial x^2}-2y\left.\overline{\frac{\partial y}{\partial x}}\right|^2\right)=\eta\psi. \quad \text{.....................}(5)$$

Using the result

$$\eta=\frac{\partial y}{\partial \psi}+\tfrac{1}{2}\psi^2\left\{\eta^2\frac{\partial^2 \eta}{\partial x^2}-2\eta\left(\frac{\partial \eta}{\partial x}\right)^2\right\},$$

and its consequence

$$\eta\psi = \psi\frac{\partial y}{\partial \psi} + \tfrac{1}{2}\left\{y^2\frac{\partial^2 y}{\partial x^2} - 2y\overline{\frac{\partial y}{\partial x}}\Big|^2\right\},$$

we can write (5) in the form

$$y - \tfrac{1}{3}\left\{y^2\frac{\partial^2 y}{\partial x^2} - 2y\overline{\frac{\partial y}{\partial x}}\Big|^2\right\} = \psi\frac{\partial y}{\partial \psi}. \quad \text{.....................(6)}$$

For the upper liquid, when we integrate the equation $\partial^2 y/\partial \psi^2 = 0$ we get

$$y - h = \eta(\psi - a), \quad \text{...........................(7)}$$

since $y = h$ when $\psi = a$.

Following the same steps as in the case of the lower liquid we get the equation

$$y - h - \tfrac{1}{3}\left\{(y-h)^2\frac{\partial^2 y}{\partial x^2} - 2(y-h)\left(\frac{\partial y}{\partial x}\right)^2\right\} = (\psi - a)\frac{\partial y}{\partial \psi}. \quad \text{......(8)}$$

Since the pressure must be continuous across the interface, we deduce from Bernouilli's pressure equation the result

$$\rho q^2 - \rho' q'^2 = (\mathrm{A} - 2gy)(\rho - \rho'), \quad \text{.... ....(9)}$$

for points on the interface, where q and q' are the velocities in the lower and upper liquids respectively at the point considered, and A is some constant.

But

$$q^2 = \frac{1 + \left(\frac{\partial y}{\partial x}\right)^2}{\left(\frac{\partial y}{\partial \psi}\right)^2},$$

and $\psi = c$, at the interface, so from (6) we find that at the interface

$$c\frac{\partial y}{\partial \psi} = y - \tfrac{1}{3}\left\{y^2\frac{\partial^2 y}{\partial x^2} - 2y\overline{\frac{\partial y}{\partial x}}\Big|^2\right\}$$

and therefore

$$q^2 = \frac{c^2\left\{1 + \left(\frac{\partial y}{\partial x}\right)^2\right\}}{y^2\left\{1 - \tfrac{1}{3}\left(y\frac{\partial^2 y}{\partial x^2} - 2\overline{\frac{\partial y}{\partial x}}\Big|^2\right)\right\}^2}$$

$$= \frac{c^2}{y^2}\left\{1 + \tfrac{2}{3}y\frac{\partial^2 y}{\partial x^2} - \tfrac{1}{3}\overline{\frac{\partial y}{\partial x}}\Big|^2\right\}, \quad \text{.....................(10)}$$

and in a similar way we find

$$q'^2 = \frac{(c-a)^2}{(y-h)^2}\left\{1 + \tfrac{2}{3}(y-h)\frac{\partial^2 y}{\partial x^2} - \tfrac{1}{3}\left(\frac{\partial y}{\partial x}\right)^2\right\}. \quad \text{.........(11)}$$

Hence (9) becomes

$$\frac{\rho c^2}{y^2} - \frac{\rho'(c-a)^2}{(y-h)^2} + \tfrac{2}{3}\left\{\frac{\rho c^2}{y} - \frac{\rho'(c-a)^2}{(y-h)}\right\}\frac{\partial^2 y}{\partial x^2} -$$

$$-\tfrac{1}{3}\left\{\frac{\rho c^2}{y^2}-\frac{\rho'(c-a)^2}{(y-h)^2}\right\}\left(\frac{\partial y}{\partial x}\right)^2=(\mathrm{A}-2gy)(\rho-\rho'), \quad (12)$$

and this is the differential equation for the form of the interface. We may write it

$$\frac{\rho c^2}{y^2}-\frac{\rho'(c-a)^2}{(y-h)^2}+\tfrac{1}{3}\left\{\frac{\rho c^2}{y}-\frac{\rho'(c-a)^2}{y-h}\right\}\frac{d}{dy}\left(\frac{dy}{dx}\right)^2$$

$$-\tfrac{1}{3}\left\{\frac{\rho c^2}{y^2}-\frac{\rho'(c-a)^2}{(y-h)^2}\right\}\left(\frac{dy}{dx}\right)^2=(\mathrm{A}-2gy)(\rho-\rho'), \quad \text{.........}(13)$$

that is,

$$\frac{d}{dy}\left[\left\{\frac{\rho c^2}{y}-\frac{\rho'(c-a)^2}{y-h}\right\}\left(\frac{dy}{dx}\right)^2\right]=3(\mathrm{A}-2gy)(\rho-\rho')-\frac{3\rho c^2}{y^2}+3\rho'\frac{(c-a)^2}{(y-h)^2}. \quad \text{............}(14)$$

Integrating this we obtain

$$\left\{\frac{\rho c^2}{y}-\frac{\rho'(c-a)^2}{y-h}\right\}\left(\frac{dy}{dx}\right)^2=3(\mathrm{A}y-gy^2)(\rho-\rho')+\frac{3\rho c^2}{y}+\frac{3\rho'(c-a)^2}{y-h}+\mathrm{D}, \quad \text{............}(15)$$

where D is a constant of integration.

Thus

$$\left(\frac{dy}{dx}\right)^2=\frac{\{\mathrm{D}+\mathrm{B}y-3g(\rho-\rho')y^2\}y(y-h)+3\rho c^2(y-h)-3\rho'(c-a)^2y}{\rho c^2(y-h)-\rho'(c-a)^2y}, \quad \text{............}(16)$$

where $3\mathrm{A}(\rho-\rho')=\mathrm{B}$.

This is the expression found by J. H. Michell for the gradient. We now assume that this expression will factorize in such a manner as to give the desired wave form, and then consider the further conditions which will make such a form possible. That is, we suppose

$$\left(\frac{dy}{dx}\right)^2=\frac{-3g(\rho-\rho')(y-k)^2(y-k_1)(y-k_2)}{\rho c^2(y-h)-\rho'(c-a)^2y}. \quad \text{............}(17)$$

This makes $dy/dx=0$ and $d^2y/dx^2=0$ when $y=k$; and $dy/dx=0$ when $y=k_1$, and when $y=k_2$.

Thus with this form the condition that the surface may be horizontal when $y=k$, is satisfied.

Now for (16) to be equivalent to (17) we must have, by equating coefficients of y,

$$k^2k_1k_2=\frac{\rho c^2h}{g(\rho-\rho')}, \quad \text{..........................}(18)$$

$$3g(\rho-\rho')\{2kk_1k_2+k^2(k_1+k_2)\}=-\mathrm{D}h+3\{\rho c^2-\rho'(c-a)^2\}, \quad ..(19)$$

$$-3g(\rho-\rho')\{k^2+2k(k_1+k_2)+k_1k_2\}=\mathrm{D}-\mathrm{B}h, \quad \text{......................}(20)$$

$$3g(\rho-\rho')\{2k+k_1+k_2\}=\mathrm{B}+3gh(\rho-\rho'), \quad \text{..........................}(21)$$

and from these equations (18)-(21) we deduce :—

$$k_1+k_2=\frac{\rho c^2}{k^2g(\rho-\rho')}-\frac{\rho'(c-a)^2}{(h-k)^2g(\rho-\rho')}+h, \quad \text{...........}(22)$$

$$\text{and } k_1 k_2 = \frac{\rho c^2 h}{k^2 g(\rho - \rho')}, \quad \text{.......................(23)}$$

so that k_1, k_2 are the roots of the equation

$$\alpha^2 - \left\{h + \frac{\rho c^2}{k^2 g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2 g(\rho-\rho')}\right\}\alpha + \frac{\rho c^2 h}{k^2 g(\rho-\rho')} = 0. \quad \text{...(24)}$$

We therefore have

$$2k_1 = h + \frac{\rho c^2}{k^2 g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2 g(\rho-\rho')}$$
$$- \sqrt{\left\{h + \frac{\rho c^2}{k^2 g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2 g(\rho-\rho')}\right\}^2 - \frac{4\rho c^2 h}{k^2 g(\rho-\rho')}},$$

$$2k_2 = h + \frac{\rho c^2}{k^2 g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2 g(\rho-\rho')}$$
$$+ \sqrt{\left\{h + \frac{\rho c^2}{k^2 g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2 g(\rho-\rho')}\right\}^2 - \frac{4\rho c^2 h}{k^2 g(\rho-\rho')}}.$$

We may write

$$\frac{c^2}{k^2} = U^2 \quad \text{and} \quad \frac{(c-a)^2}{(h-k)^2} = V^2,$$

since U is the velocity at infinity of the undisturbed lower liquid of depth k, and V is the velocity at infinity of the undisturbed upper liquid of depth $(h-k)$.

If we also write $\rho' = \lambda\rho$ and $V^2 = \mu U^2$ we have :—

$$2k_1 = h + \frac{U^2}{g(1-\lambda)}(1-\lambda\mu) - \sqrt{\left\{h + \frac{U^2(1-\lambda\mu)}{g(1-\lambda)}\right\}^2 - \frac{4hU^2}{g(1-\lambda)}},$$

$$2k_2 = h + \frac{U^2}{g(1-\lambda)}\left\{1-\lambda\mu\right\} + \sqrt{\left\{h + \frac{U^2(1-\lambda\mu)}{g(1-\lambda)}\right\}^2 - \frac{4hU^2}{g(1-\lambda)}}.$$

Necessary Conditions for such a Wave.

We have put the equation for the gradient into the form

$$\left(\frac{dy}{dx}\right)^2 = \frac{3g(1-\lambda)(y-k)^2(y-k_1)(y-k_2)}{U^2[k^2 h - \{k^2 - \lambda\mu(h-k)^2\}y]}.$$

Now the denominator may be written $k^2(h-y) + \lambda\mu(h-k)^2 y$ and y is less than h at all points on the interface. Therefore the denominator is always positive. Hence, assuming $\lambda < 1$ (*i.e.*, $\rho' < \rho$), we must have $y - k_1$ and $y - k_2$ of the same sign, to make dy/dx real.

But y lies between k and either k_1 or k_2, since k, k_1, and k_2 are the turning values of y. Therefore, either

(i) $k < y < k_1 < k_2$,

or (ii) $k > y > k_2 > k_1$.

These alternatives represent

(i) a crested wave,

or (ii) an inverted wave.

There is no wave for a value of k between k_1 and k_2.

Thus, for values of k between 0 and k_1 there is a crested wave, and for values of k between k_2 and h there is an inverted wave.

Lamb has treated the infinitesimal wave at the interface between two liquids (see Lamb's Hydrodynamics, Arts. 231-234), and if in Lamb's result we make the wave length tend to infinity, we find, as we should expect, that the two heights at which infinitesimal long waves are possible are k_1 and k_2.

Now, since k_1 will be the height of the crest when a crested wave exists and k_2 will be the depth of the lower liquid at the trough in the case of an inverted wave, it will be necessary for k_1 and k_2 to be real if there is to be a wave form at all. Therefore, referring to the equation (24), we deduce the condition

$$\left\{h + \frac{U^2(1-\lambda\mu)}{g(1-\lambda)}\right\}^2 \nless \frac{4hU^2}{g(1-\lambda)}.$$

Approximation-Method for High Waves.

If we take k very small we find approximately

$$\left(\frac{dy}{dx}\right)^2 = C\frac{y(y-2k)(y-k_1)(y-k_2)}{y}$$
$$= C(y-2k)(y-k_1)(y-k_2),$$

for values of y near the crest, where C is a known constant. This makes

$$\sqrt{C}\,x = \int\frac{dy}{\sqrt{(y-2k)(y-k_1)(y-k_2)}}.$$

Hence we can find an approximate form for the wave in terms of an elliptic integral when the wave is near its greatest height.

The Asymmetric Long Wave.

There is, however, a type of long wave whose form can be determined from the differential equation without further approximation. This is the wave which we get on putting $k_1 = k_2$. Its differential equation is

$$\left(\frac{dy}{dx}\right)^2 = \frac{3g(1-\lambda)(y-k)^2(k_1-y)^2}{U^2[k^2h - \{k^2-\lambda\mu(h-k)^2\}y]},$$

and therefore when $y=k$, or $y=k_1$, $dy/dx=0$, and $d^2y/dx^2=0$.

This means that the wave has no crests but rises gradually, through an infinite horizontal distance from $y=k$ to $y=k_1$. The motion is here of the nature of a "bore."

Since k_1 and k_2 are the roots of equation (24), the condition that k_1 should be equal to k_2 is

$$\left\{\frac{\rho c^2}{h^2g(\rho-\rho')} - \frac{\rho'(c-a)^2}{(h-k)^2g(\rho-\rho')} + h\right\}^2 = \frac{4\rho c^2h}{g(\rho-\rho')h^2}, \quad \ldots\ldots\ldots(25)$$

that is,

$$\left\{\frac{U^2}{g(1-\lambda)}-\frac{V^2\lambda}{g(1-\lambda)}+h\right\}^2=\frac{4U^2h}{g(1-\lambda)}. \quad \text{.................(26)}$$

This gives

$$h^2-\frac{2}{g(1-\lambda)}(U^2+\lambda V^2)h+\frac{(U^2-\lambda V^2)^2}{g^2(1-\lambda)^2}=0, \quad \text{...........(27)}$$

and the roots of this are always real, since $(U^2+\lambda V^2)^2>(U^2-\lambda V^2)^2$. This means that for any given pair of values of the velocities U, V, of the currents, there are two possible values of h, the distance apart of the horizontal boundaries; they are given by

$$h=\frac{1}{g(1-\lambda)}(U\pm\sqrt{\lambda}V)^2. \quad \text{...................(28)}$$

If we regard equation (26) as an equation for V in terms of h and U we find

$$|V|=\frac{U\pm\sqrt{gh(1-\lambda)}}{\sqrt{\lambda}} \quad \text{.......................(29)}$$

When condition (26) is satisfied, we have, from (22)

$$k_1=k_2=\tfrac{1}{2}\left\{h+\frac{U^2}{g(1-\lambda)}-\frac{V^2\lambda}{g(1-\lambda)}\right\}, \quad \text{.................(30)}$$

and on substituting for V from equation (29) we deduce

$$k_1=h\sqrt{\frac{U^2}{gh(1-\lambda)}}. \quad \text{.......................(31)}$$

The positive sign with the root in (29) would give

$$k_1=-h\sqrt{\frac{U^2}{gh(1-\lambda)}},$$

and we consider, therefore, only the negative sign. That is, we take

$$\lambda V^2=U^2\left\{1-\frac{\sqrt{(1-\lambda)gh}}{U}\right\}^2$$

$$=U^2\left(\frac{h}{k_1}-1\right)^2.$$

Now, returning to the equation for $(dy/dx)^2$, these results give

$$(dy/dx)^2=\frac{3g(1-\lambda)}{U^2}\;\frac{(y-k)^2(k_1-y)^2}{[hk^2-\{k^2-(h-k)^2(1-h/k_1)^2\}y]}$$

$$=\frac{3g(1-\lambda)k_1^2}{U^2h}\;\frac{(y-k)^2(k_1-y)^2}{[k^2k_1^2-\{k_1^2-(h-k)^2(h-k_1)^2\}y/h]}$$

$$=\frac{3(y-k)^2(k_1-y)^2}{[k^2k_1^2-\{k^2k_1^2-(h-k)^2(h-k_1)^2\}y/h]}.$$

Three cases now arise, depending on whether

(i) $kk_1 = (h-k)(h-k_1)$,

(ii) $kk_1 > (h-k)(h-k_1)$,

(iii) $kk_1 < (h-k)(h-k_1)$.

We shall now consider these separately.

(i) Here we have $kk_1 = (h-k)(h-k_1)$
and, therefore, $k+k_1 = h$.
This means that the highest and lowest levels of the wave are equidistant from the mean height $h/2$ of the liquids.
In this case

$$\left(\frac{dy}{dx}\right)^2 = \frac{3(y-k^2)(k_1-y^2)}{k^2k_1^2},$$

and therefore

$$\frac{dy}{dx} = \frac{\sqrt{3}}{kk_1}(y-k)(k_1-y),$$

where y lies between k and k_1.

On integrating, if we choose the origin so that the constant of integration is zero, we find

$$\frac{\sqrt{3}}{kk_1}x = \int \frac{dy}{\left(\frac{k_1-k}{2}\right)^2 - \left(y - \frac{k+k_1}{2}\right)^2}$$

$$= \frac{2}{k_1-k} \text{ artanh } \frac{y - \frac{k+k_1}{2}}{\frac{k_1-k}{2}}.$$

Now changing over to a horizontal axis along the mean level, so that $y' = y - \frac{k+k_1}{2}$, we find the equation to the wave form is

$$\frac{k_1-k}{kk_1}\,\frac{\sqrt{3}}{2}x = \text{artanh}\frac{2y'}{k_1-k},$$

that is, $y' = a \tanh mx$,

where $a = \frac{k_1-k}{2}$,

$$m = \frac{\sqrt{3}}{2}\left(\frac{1}{k} - \frac{1}{k_1}\right).$$

(ii) Consider now the case $kk_1 > (h-k)(h-k_1)$.
Here

$$\left(\frac{dy}{dx}\right)^2 = \frac{3(y-k)^2(k_1-y)^2}{[k^2k_1^2 - \{k^2k_1^2 - (h-k)^2(h-k_1)^2\}y/h]}$$

$$= \frac{\alpha^2(y-k)^2(k_1-y)^2}{\beta^2 - y},$$

where $\alpha^2 = \dfrac{3h}{k^2k_1^2 - (h-k)^2(h-k_1)^2}$ and $\beta^2 = \dfrac{k^2k_1^2}{3}\alpha^2.$

Now, since $kk_1 > (h-k)(h-k_1)$,
therefore $h < k + k_1$,
and therefore $h - k < k_1$ and $h - k_1 < k$.

But
$$\beta^2 = \frac{k^2k_1^2h}{k^2k_1^2 - (h-k)^2(h-k_1)^2}$$
$$= \frac{h}{1 - \dfrac{(h-k)^2(h-k_1)^2}{k^2k_1^2}}$$
$$= \frac{h}{1-\theta}, \text{ where } 0 < \theta < 1.$$

Therefore $\beta^2 > h$,
$> k_1$,
$> k$.

Now $\dfrac{dx}{dy} = \dfrac{1}{\alpha}\dfrac{\sqrt{\beta^2 - y}}{(y-k)(k_1-y)}$, therefore

$$(k_1 - k)\alpha x = \int \frac{\sqrt{\beta^2 - y}}{y-k}\,dy + \int \frac{\sqrt{\beta^2 - y}}{k_1 - y}\,dy$$
$$= -2\sqrt{\beta^2 - k}\ \text{artanh}\ \frac{\sqrt{\beta^2 - y}}{\sqrt{\beta^2 - k}} + 2\sqrt{\beta^2 - k_1}\ \text{arcoth}\ \frac{\sqrt{\beta^2 - y}}{\sqrt{\beta^2 - k_1}},$$

which is the equation to the wave form in this second case.

(iii) If $kk_1 < (h-k)(h-k_1)$,

$$(dy/dx)^2 = \frac{\alpha^2(y-k)^2(y-k_1)^2}{\beta^2 + y},$$

where $\alpha^2 = \dfrac{3h}{(h-k)^2(h-k_1)^2 - k^2k_1^2}$,

and $\beta^2 = \dfrac{k^2k_1^2}{3}\alpha^2.$

Therefore $\dfrac{dx}{dy} = \dfrac{1}{\alpha}\dfrac{\sqrt{\beta^2 + y}}{(y-k)(k_1-y)}$, and

$$\alpha x = \int \frac{\sqrt{\beta^2 + y}}{(y-k)(k_1 - y)}\,dy$$
$$= \int \sqrt{\beta^2 + y}\left\{\frac{1}{y-k} + \frac{1}{k_1 - y}\right\}\frac{1}{k_1 - k}\,dy.$$

Therefore

$$(k_1-k)\alpha x=2\sqrt{k_1+\beta^2}\ \operatorname{artanh}\frac{\sqrt{y+\beta^2}}{\sqrt{k_1+\beta^2}}-2\sqrt{k+\beta^2}\ \operatorname{arcoth}\frac{\sqrt{y+\beta^2}}{\sqrt{k+\beta^2}}.$$

The present paper embodies a small portion of the work done under the terms of a Research Grant for Mathematics in the University of Melbourne. The publication of the rest of the work has been postponed owing to the author's departure for England.